Inner Product Space

Linear Algebra

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu

Maryam Ramezani maryam.ramezani@sharif.edu

Overview

- Linear Form
- Bilinear Form
- Bilinear Form on Complex Vector Space
- Inner Product
- Inner Product Space

Linear Form

What are Linear Functions?

- $f: R^n \to R$ means that f is a function that maps real $n-$vectors to real numbers
- $f(x)$ is the value of function f at x (x is referred to as the argument of the function).
- $f(x) = (x_1, x_2, \ldots, x_n)$: argument

Definition

A function $f: R^n \to R$ is linear if it satisfies the following two properties:

- Additivity: For any $n-$vector x and y, $f(x + y) = f(x) + f(y)$
- Homogeneity: For any $n-$vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$

Superposition property:

Definition

Superposition property:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Note

- ❑ A function that satisfies the superposition property is called linear

Homogeneity and Additivity

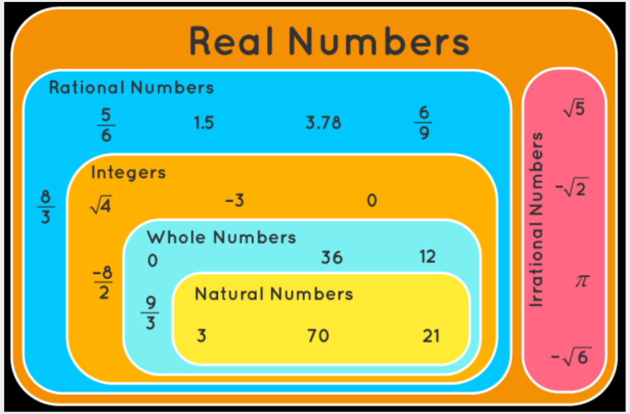
Definition

- Additivity:

For any n−vector x and y, $f(x+y) = f(x) + f(y)$

- Homogeneity:

For any n−vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$



Counterexample:
$f(x) = \sqrt{2}x$

What are Linear Functions?

- If a function f is linear, superposition extends to linear combinations of any number of vectors:

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

Inner product is Linear Function?

Theorem

A function defined as the inner product of its argument with some fixed vector is linear.

Proof?

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

What are Linear Functions?

Theorem

If a function is linear, then it can be expressed as the inner product of its argument with some fixed vector.

Proof?

What are Linear Functions?

Theorem

The representation of a linear function f as $f(x) = a^T x$ is unique, which means that there is only one vector a for which $f(x) = a^T x$ holds for all x.

Proof?

Linear Form Examples

Example

- Is average a linear function?
- Is maximum a linear function?

Bilinear Form

Definition

Suppose V and W are vector spaces over the same field $\mathbb{F}$. Then a function $f: V \times W \rightarrow \mathbb{F}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:
   i. $f(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = f(\mathbf{v}_1, \mathbf{w}) + f(\mathbf{v}_2, \mathbf{w})$ and
   ii. $f(c\mathbf{v}_1, \mathbf{w}) = cf(\mathbf{v}_1, \mathbf{w})$ for all $c \in \mathbb{F}, \mathbf{v}_1, \mathbf{v}_2 \in V$, and $\mathbf{w} \in W$.

b) It is linear in its second argument:
   i. $f(\mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2) = f(\mathbf{v}, \mathbf{w}_1) + f(\mathbf{v}, \mathbf{w}_2)$ and
   ii. $f(\mathbf{v}, c\mathbf{w}_1) = cf(\mathbf{v}, \mathbf{w}_1)$ for all $c \in \mathbb{F}, \mathbf{v} \in V$, and $\mathbf{w}_1, \mathbf{w}_2 \in W$.

Note

Let V be a vector space over a field $\mathbb{F}$. Then the **dual** of V, denoted by V^*, is the vector space consisting of all linear forms on V.

Example

Let V be a vector space over a field $\mathbb{F}$. Show that the function $g: V^* \times V \rightarrow \mathbb{F}$ defined by

$$g(f, \mathbf{v}) = f(\mathbf{v}) \text{ for all } f \in V^*, \mathbf{v} \in V$$

is a bilinear form.

Positive Definite Bilinear Form

Definition

A **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V is called positive definite if for all $v \in V, v \neq 0$:

$$f(v, v) > 0$$

Example

Which one is a positive definite bilinear form?

- ❑ $f(x, y) = x_1 y_1 - 2x_1 y_2 - 2x_2 y_1 + 5x_2 y_2$
- ❑ $f(x, y) = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 3x_2 y_2$

Symmetric Bilinear Form

Definition

A **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V is called symmetric if for all $v, w \in V$:

$$f(v, w) = f(w, v)$$

Bilinear Form arises from a matrix

Theorem

Every **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V arises from a matrix for all $v, w \in V$:

$$f(v, w) = v^T A w$$

Proof?

Associated Matrices

Definition

If V is a finite-dimensional vector space, $B = \{b_1, \ldots, b_n\}$ is a basis of V, and $f: V \times V \rightarrow \mathbb{F}$ be a **bilinear form** function the associated matrix A of f with respect to B is the matrix $[f]_B \in \mathbb{F}^{n \times n}$ whose (i, j)-entry is the value $f(b_i, b_j)$.

$$f(v, w) = v^T A w = v^T [f]_B w$$

$$[f]_{\mathcal{B}} = \begin{pmatrix} f(b_1, b_1) & \ldots & f(b_1, b_n) \\ \vdots & & \vdots \\ f(b_n, b_1) & \ldots & f(b_n, b_n) \end{pmatrix}$$

Note

The associated matrix changes if we use a different basis.

Example

For the bilinear form $f\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = 2ac + 4ad - bc$ on $\mathbb{F}^2$, find $[f]_B$ for basis $B = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}\}$ and $[f]_P$ for basis P= $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$

Bilinear Form
Over Complex Vector Space

Bilinear Form over a complex vector space

Definition

Suppose V and W are vector spaces over the same field $\mathbb{C}$. Then a function $f: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:
   i. $f(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = f(\mathbf{v}_1, \mathbf{w}) + f(\mathbf{v}_2, \mathbf{w})$ and
   ii. $f(\lambda \mathbf{v}_1, \mathbf{w}) = \lambda f(\mathbf{v}_1, \mathbf{w})$ for all $\lambda \in \mathbb{C}, \mathbf{v}_1, \mathbf{v}_2 \in V$, and $\mathbf{w} \in W$.

b) It is conjugate linear in its second argument:
   i. $f(\mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2) = f(\mathbf{v}, \mathbf{w}_1) + f(\mathbf{v}, \mathbf{w}_2)$ and
   ii. $f(\mathbf{v}, \lambda \mathbf{w}_1) = \bar{\lambda} f(\mathbf{v}, \mathbf{w}_1)$ for all $\lambda \in \mathbb{C}, \mathbf{v} \in V$, and $\mathbf{w}_1, \mathbf{w}_2 \in W$.

Bilinear forms (real vs complex vector space)

Bilinear forms on $\mathbb{R}^n$	Bilinear forms on $\mathbb{C}^n$
Linear in the first variable	Linear in the first variable
Linear in the second variable	Conjugate linear in the second variable

Inner product

Definition

An inner product is a positive-definite symmetric bilinear form.

- An inner product on V is a function $\langle , \rangle : V \times V \rightarrow \mathbb{R}$ such that $v, w \in V, c \in \mathbb{R}$:

1. $\langle v, v \rangle = 0$ if and only if $v = 0$.
2. $\langle w, v \rangle = \langle v, w \rangle$.
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.
4. $\langle cw, u \rangle = c\langle w, u \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, v \rangle \geq 0$ for all $v \in V$.

Why for bilinear form I wrote just two properties instead of four properties?

- Using properties (2) and (4) and again (2)

$$\langle w, cu\rangle = \langle cu, w\rangle = c\langle u, w\rangle = c\langle w, u\rangle$$

- Using properties (2), (3) and again (2)

$$\langle w, u+v\rangle = \langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle = \langle w, u\rangle + \langle w, v\rangle$$

1. $\langle v, v\rangle = 0$ if and only if $v = 0$.
2. $\langle w, v\rangle = \langle v, w\rangle$.
3. $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all u, $v, w \in V$.
4. $\langle cw, u\rangle = c\langle w, u\rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.
5. $\langle v, v\rangle \geq 0$ for all $v \in V$.

Inner Products

Note

- For $v \in V$, $\langle 0, v\rangle = 0$, $\langle v, 0\rangle$ =0.

General Inner product

Definition

Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and that V is a vector space over $\mathbb{F}$. Then an **inner product** on V is a function $\langle \cdot,\cdot \rangle : V \times V \rightarrow \mathbb{F}$ such that the following three properties hold for all $c \in \mathbb{F}$ and all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$:

a) $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ (conjugate symmetry)

b) $\langle \mathbf{v}+c\mathbf{x}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + c\langle \mathbf{x}, \mathbf{w} \rangle$ (linearity)

c) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$. (pos. definiteness)

Inner Products for vectors

Note

- The standard inner product between vectors is: $(x, y \in \mathbb{R}^n)$

$$\langle x, y\rangle = x^T y = \sum x_i y_i$$

- The function $\langle \cdot,\cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ defined by

$$\langle \mathrm{v}, \mathrm{w}\rangle = v^* \mathrm{w} = \sum_{i=1}^{n} \overline{v_i}\, w_i$$

for all $\mathrm{v}, \mathrm{w} \in \mathbb{C}^n$ is an inner product on $\mathbb{C}^n$.

Inner Product for matrices

Note

❑ The standard inner product between two matrices is: $(X, Y \in \mathbb{R}^{m\times n})$

$$\langle X, Y\rangle = trace(X^T Y) = \sum_i \sum_j X_{ij} Y_{ij}$$

Example

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Inner Product for functions

Note

- Let $a < b$ be real numbers and let $C[a, b]$ be the vector space of continuous functions on the real interval $[a, b]$. The function $\langle \cdot,\cdot \rangle : C[a, b] \times C[a, b] \to \mathbb{R}$ defined by

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx \qquad \text{for all} \qquad f, g \in C[a, b]$$

is and inner product on $C[a, b]$.

Inner Product for polynomials

Note

- For $p(x)$ and $q(x)$ with at most degree n:

$$\langle p(x), q(x)\rangle = \mathrm{p}(0)\mathrm{q}(0) + \mathrm{p}(1)\mathrm{q}(1) + \cdots + \mathrm{p}(\mathrm{n})\mathrm{q}(\mathrm{n})$$

- For $p(x)$ and $q(x)$: $\langle p(x), q(x)\rangle = \mathrm{p}(0)\mathrm{q}(0) + \int_{-1}^{1} p'q'$

- For $p(x)$ and $q(x)$: $\langle p(x), q(x)\rangle = \int_{0}^{\infty} p(x)q(x)e^{-x}dx$

Definition

An inner product space is a finite-dimensional real or complex vector space V along with an inner product on V.

real → Euclidean Space

complex → Unitary Space

References

- Chapter 8: Kenneth Hoffman and Ray A. Kunze. Linear Algebra. PHI Learning, 2004.
- Chapter 6: Sheldon Axler, Linear Algebra Done Right, 2024.
- Chapter 1: David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Pearson, 2016.
- Chapter 2: David Poole, Linear Algebra: A Modern Introduction. Cengage Learning, 2014.
- Chaper1: Gilbert Strang. Introduction to Linear Algebra. Wellesley-Cambridge Press, 2016.